Power grid vulnerability, new models, algorithms and computing

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July, 2009

The N-k problem in power grids

Given a power grid modeled by a network, delete a **small** set of arcs, such that in the resulting network all feasible flows have **small** throughput

- Used to model "natural" blackouts
- "Small" throughput: we satisfy less than some amount D^{min} of total demand
- "Small" set of arcs = **very** small
- Delete **1** arc = the "N-1" problem
- Of interest: delete $k = 2, 3, 4, \ldots$ edges
- Naive enumeration blows up

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Linear power flow model

We are given a network G with:

- A set of S of **supply** nodes (the "generators"); for each generator i an "operating range" $0 \leq S_i^L \leq S_i^U$,
- A set D of **demand** nodes (the "loads"); for each load i a "maximum demand" $0 \leq D_i^{max}$.
- For each arc (i, j) values x_{ij} and u_{ij}.

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Feasible power flows

A **power flow** is a solution f, θ to:

- $\sum_{ij} f_{ij} - \sum_{ij} f_{ji} = b_i$, for all i, where

  $S_i^L \leq b_i \leq S_i^U$ **OR** $b_i = 0$, for each $i \in S$,

  $0 \leq -b_i \leq D_i^{max}$ for $i \in D$,

  and $b_i = 0$, otherwise.

- $x_{ij} f_{ij} - \theta_i + \theta_j = 0$ for all (i, j). (Ohm's equation)

Lemma Given a choice for b with $\sum_i b_i = 0$, the system has a **unique** solution.

The solution is **feasible** if $|f_{ij}| \leq u_{ij}$ for every (i, j).

Its **throughput** is $\sum_{i \in D} -b_i$.

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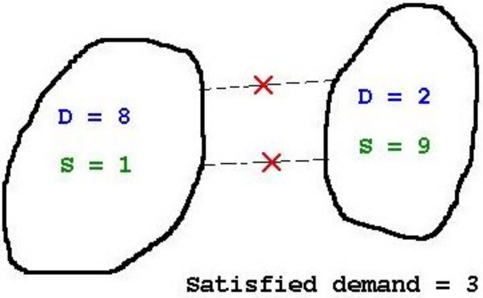
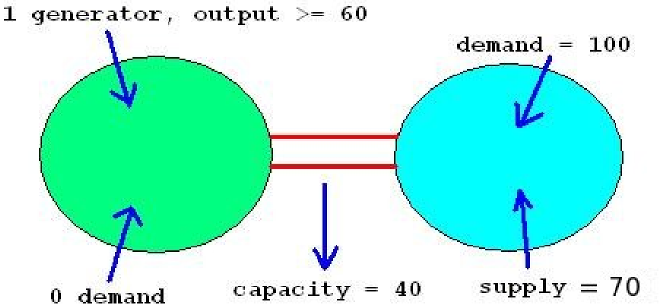
Three types of successful attacks

Type 1: Network becomes disconnected with a mismatch of supply and demand.



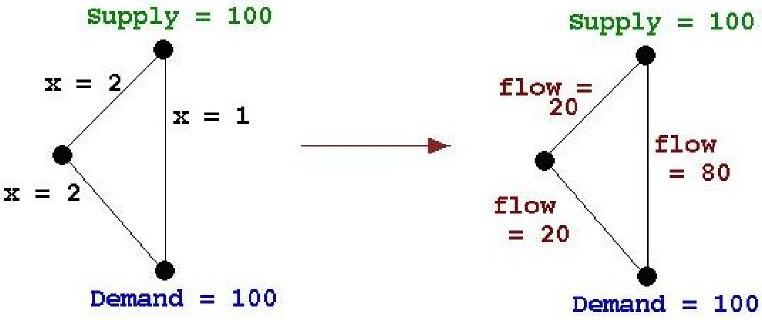
Three types of successful attacks

Type 2: Lower bounds on generator ouptuts cause line overload



Three types of successful attacks

Type 3: Uniqueness of power flows means exceeded capacities or insufficient supply.



A game:

The *controller's problem:* Given a set $\mathcal{A}$ of arcs that has been deleted by the attacker, choose a set $\mathcal{G}$ of generators to operate, so as to feasibly meet demand (at least) D^{min}.

The *attacker's problem:* Choose a set $\mathcal{A}$ of arcs to delete, so as to defeat the controller, no matter how the controller chooses $\mathcal{G}$.

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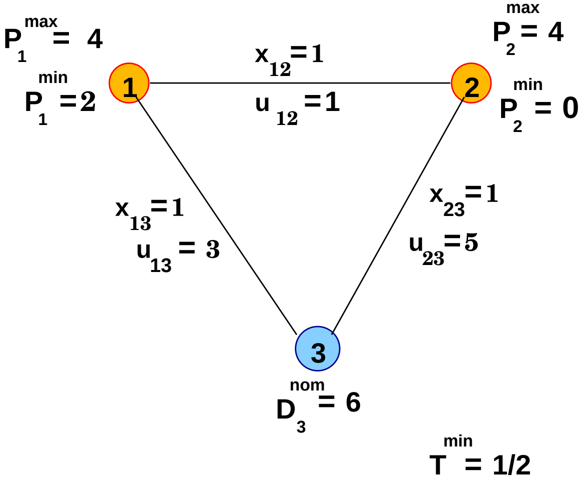
$P_1^{max} = 4$
$P_1^{min} = 2$
$x_{12} = 1$
$u_{12} = 1$
$P_2^{max} = 4$
$P_2^{min} = 0$
1
2
3
$x_{13} = 1$
$u_{13} = 3$
$x_{23} = 1$
$u_{23} = 5$
$D_3^{nom} = 6$
$T^{min} = 1/2$

The controller's problem for a given choice of generators

Given a set $\mathcal{A}$ of arcs that has been deleted by the attacker, **AND** a choice $\mathcal{G}$ of which generators to operate, set demands and supplies so as to feasibly meet total demand (at least) D^{min}.

This a linear program:

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This a linear program:

$t_{\mathcal{A}}(\mathcal{G}) \doteq \min t$

Subject to:

$\sum_{ij} f_{ij} - \sum_{ij} f_{ji} - b_i = 0$, for all nodes i,

$S_i^{min} \leq b_i \leq S_i^{max}$ for $i \in \mathcal{G}$, $\quad 0 \leq -b_i \leq D_i^{max}$ for $i \in D$

$b_i = 0$ otherwise.

$x_{ij} f_{ij} - \theta_i + \theta_j = 0$ for all $(i,j) \notin \mathcal{A}$

$-\sum_{i \in D} b_i + D^{\min} t \geq 2D^{\min}$

$u_{ij} t \geq |f_{ij}|$ for all $(i,j) \notin \mathcal{A}$

$f_{ij} = 0$ for all $(i,j) \in \mathcal{A}$

Lemma: $t_{\mathcal{A}}(\mathcal{G}) > 1$ iff the attack is successful against the choice $\mathcal{G}$.

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for all $(i,j) \in \mathcal{A}$, $t \geq 1 + |f_{ij}|/u_{ij}$

Lemma: $t_{\mathcal{A}}(\mathcal{G}) > 1$ iff the attack is successful against the choice $\mathcal{G}$.

Attack problem

$\min \sum_{ij} z_{ij}$

Subject to:

$z_{ij} = 0$ or 1, for all arcs (i,j), (choose which arcs to delete)

$t_{suppt(z)}(\mathcal{G}) > 1$, for every subset $\mathcal{G}$ of generators.

[$suppt(v) =$ support of v]

$\longrightarrow$ Use LP dual to represent $t_{suppt(z)}(\mathcal{G})$

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$-\left(\sum_{i \in D} b_i\right) / D^{min} + t \geq 2$

$u_{ij} t \geq |f_{ij}|$ for all $(i, j) \notin \mathcal{A}$ (p_{ij}, q_{ij})

$u_{ij} t \geq u_{ij} + |f_{ij}|$ for all $(i, j) \in \mathcal{A}$ (r_{ij}^{+}, r_{ij}^{-})

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0-1 -ify: form mip-dual

$$p_{ij} + q_{ij} \le M_{ij}(1 - z_{ij})$$

$$r^+_{ij} + r^-_{ij} \le M'_{ij} z_{ij}$$

$\rightarrow$ "big M" formulation: what's the problem

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I hate math

$$M_{ij} \;=\; \sqrt{x_{ij}}\,\max_{(k,l)}\left(\sqrt{x_{kl}}\;u_{kl}\right)^{-1}$$

A formulation for the attack problem

$\min \sum_{ij} z_{ij}$

Subject to:

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$\longrightarrow$ very large

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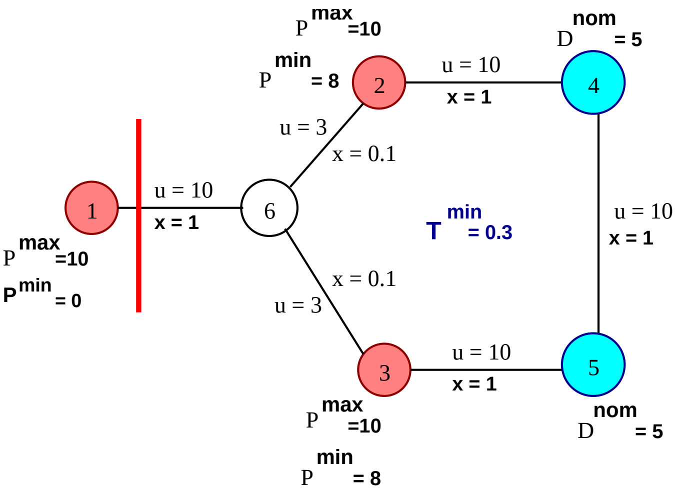
P max =10
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x = 1
nom
D = 5
1
2
4
6
u = 3
x = 0.1
u = 10
x = 1
T min = 0.3
u = 10
x = 1
P max =10
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3
5
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P min = 8
nom
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$\longrightarrow$ Maintain a “master (attacker) MIP”:

- Made up of valid inequalities (for the attacker)
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Iterate:

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Cutting planes = Benders' cuts

For a given $\mathbf{0}-\mathbf{1}$ vector $\hat{\boldsymbol{z}}$, and a set of generators $\mathcal{G}$,

$$t_{suppt(\hat{z})}(\mathcal{G}) \;=\; \max \mu^T y$$

s.t.

$$Ay \;\leq\; b\hat{z}$$

$$y \;\in\; P$$

for some vectors μ, $\boldsymbol{b}$, matrix $\boldsymbol{A}$ and polyhedron $\boldsymbol{P}$,
(all dependent on $\mathcal{G}$, but not $\hat{\boldsymbol{z}}$).

$\rightarrow$ If $t_{suppt(\hat{z})}(\mathcal{G}) \leq 1$, use LP duality to separate $\hat{z}$,

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Plus:

Given an **unsuccessful** attack z^*,

"Pad" it: choose arcs $a_1, a_2, \ldots, a_k$ such that

$$supp(z^*) \cup \{a_1, a_2, \ldots, a_{k-1}, a_k\} \text{ is successful, but}$$

$$supp(z^*) \cup \{a_1, a_2, \ldots, a_{k-1}\} \text{ **is not**}$$

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Plus, combinatorial relaxations

Strengthen controller or weaken attacker $\longrightarrow$ obtain valid attacks (e.g. upper bounds)

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Strengthen attacker or weaken controller $\longrightarrow$ obtain valid lower bounds.

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IEEE 57 nodes, 78 arcs, 4 generators Entries show: (iteration count), CPU seconds, Attack status (**F** = cardinality too small, **S** = attack success)					
	Attack cardinality				
Min. thrpt	**2**	**3**	**4**	**5**	**6**
0.75	(1), 2, **F**	(2), 3, **S**			
0.70	(1), 1, **F**	(3), 7, **F**	(48), 246, **F**	(51), 251, **S**	
0.60	(2), 2, **F**	(3), 6, **F**	(6), 21, **F**	(6), 21, **S**	
0.50	(2), 2, **F**	(3), 7, **F**	(6), 13, **F**	(6), 13, **F**	(6), 13, **S**
0.30	(1), 1, **F**	(2), 3, **F**	(2), 3, **F**	(2), 3, **F**	(2), 3, **F**

Table: ***IEEE 57-bus test case***

118 nodes, 186 arcs, 17 generators Entries show: (iteration count), CPU seconds, Attack status (**F =** cardinality too small, **S** = attack success)			
	Attack cardinality		
Min.	**2**	**3**	**4**
thrpt			
0.92	(4), 18, **S**		
0.90	(5), 180, **F**	(6), 193, **S**	
0.88	(4), 318, **F**	(6), 595, **S**	
0.84	(2), 23, **F**	(6), 528, **F**	(148), 6562, **S**
0.80	(2), 18, **F**	(5), 394, **F**	(7), 7755, **F**
0.75	(2), 14, **F**	(4), 267, **F**	(7), 6516, **F**

Table: ***IEEE 118-bus test case***

98 nodes, 204 arcs Entries show: (iteration count), time, Attack status (**F =** cardinality too small, **S** = attack success)			
12 generators			
	Attack cardinality		
Min. throughput	**2**	**3**	**4**
0.92	(2), 318, **F**	(11), 7470, **F**	(14), 11819, **S**
0.90	(2), 161, **F**	(11), 14220, **F**	(18), 16926, **S**
0.88	(2), 165, **F**	(10), 11178, **F**	(15), 284318, **S**
0.84	(2), 150, **F**	(9), 4564, **F**	(16), 162645, **F**
0.75	(2), 130, **F**	(9), 7095, **F**	(15), 93049, **F**

98 nodes, 204 arcs Entries show: (iteration count), time, Attack status (**F** = cardinality too small, **S** = attack success)			
15 generators			
	Attack cardinality		
Min. throughput	**2**	**3**	**4**
0.94	(2), 223, **F**	(11), 654, **S**	
0.92	(2), 201, **F**	(11), 10895, **F**	(18), 11223, **S**
0.90	(2), 193, **F**	(11), 6598, **F**	(16), 206350, **S**
0.88	(2), 256, **F**	(9), 15445, **F**	(18), 984743, **F**
0.84	(2), 133, **F**	(9), 5565, **F**	(15), 232525, **F**
0.75	(2), 213, **F**	(9), 7550, **F**	(11), 100583, **F**

Min. Throughput	Min. Attack Size	Time (sec.)
0.95	2	2
0.90	3	20
0.85	4	246
0.80	5	463
0.75	6	2158
0.70	6	1757
0.65	7	3736
0.60	7	1345
0.55	8	2343
0.50	8	1328

Table: 49 nodes, 84 arcs, one configuration

A different model

What are we looking for? **“Hidden”**, **“small”**, **“counterintuitive”** weaknesses of a grid.

→ The expectation is that such weaknesses exist, and we need a method to reveal them

→ Allow the adversary to selectively place stress on the grid in order to cause failure

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Power flows (again)

A **power flow** is a solution f, θ to:

- $\sum_{ij} f_{ij} - \sum_{ij} f_{ji} = b_i$, for all i, where

  $b_i > 0$ when i is a generator,

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  and $b_i = 0$, otherwise.

- $x_{ij} f_{ij} - \theta_i + \theta_j = 0$ for all (i, j).

Lemma Given a choice for b with $\sum_i b_i = 0$, the system has a **unique** solution.

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Model

(I) The attacker *sets* the resistance x_{ij} of any arc (i, j).

(II) The attacker is constrained: we must have $x \in F$ for a certain known set F.

(III) The output of each generator i is fixed at a given value P_i, and similarly each demand value D_i is also fixed at a given value.

(IV) The objective of the attacker is to maximize the overload of any arc, that is to say, the attacker wants to solve

$$\max_{x \in F} \max_{ij} \left\{ \frac{|f_{ij}(x)|}{u_{ij}} \right\},$$

Example for F :

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Lemma (excerpt)

Let S be a set of arcs whose removal does not disconnect G.

Suppose we set $\boldsymbol{x_{st} = L}$ for each arc $(s, t) \in S$.

Let $\boldsymbol{f(x)}$ denote the resulting power flow, and let $\boldsymbol{\bar{f}}$ the solution to the power flow problem on $\boldsymbol{G - S}$.

Then

(a) $\lim_{L \to +\infty} f_{st}(x) \; = \; 0$, for all $(s, t) \in S$,

(b) For any $(u, v) \notin S$, $\lim_{L \to +\infty} f_{uv}(x) \; = \; \bar{f}_{uv}$.

How to solve the problem

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Smooth version:

$$\begin{aligned} \max_{x,p} \quad & \sum_{ij} \frac{f_{ij}(x)}{u_{ij}}(p_{ij} - q_{ij}) \\ \text{s.t.} \quad & \sum_{ij}(p_{ij} + q_{ij}) = 1, \\ & x \in F, \quad p, q \geq 0. \end{aligned}$$

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Methodology

→ A recent research trend: adapt methodologies from **smooth**, convex optimization to **smooth**, non-convex optimization.

→ Several industrial-strength codes.

Our objective:

$$F(x,p) \;=\; \sum_{ij} \frac{f_{ij}(x)}{u_{ij}}(p_{ij} - q_{ij})$$

Lemma: There exist efficient, sparse linear algebra algorithms for computing the gradient $\nabla_{x,p}F(x,p)$ and Hessian $\frac{\partial^2 F(x,p)}{\partial^2 x,p}$

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Some details

Implementation using LOQO (currently testing SNOPT)

Adversarial model:

$$\sum_{ij} x_{ij} \le B, \qquad x_{ij}^L \le x_{ij} \le x_{ij}^U \quad \forall\,(i,j),$$

where (this talk):

$$x_{ij}^L = 1, \quad x_{ij}^U = 10, \quad \forall\,(i,j),$$

and

$$\sum_{(i,j)} x_{ij} = \sum_{(i,j)} x_{ij}^L + \Delta\mathrm{B},$$

where

$$\Delta\mathrm{B} \le 40$$

Sample computational experience

Table: ***57 nodes, 78 arcs***

Iteration Limit: 700, $\epsilon = 0.01$

	ΔB 9	18	27	36
Max Cong	1.070	1.190	1.220	1.209
Time (sec)	8	19	19	19
Iterations	339	Limit	Limit	Limit
Exit Status	ϵ-L-opt.	PDfeas. Iter: 700	PDfeas. Iter: 700	PDfeas. Iter: 700

Sample computational experience

Table: *118 nodes, 186 arcs*

Iteration Limit: 700, $\epsilon = 0.01$

	ΔB			
	9	**18**	**27**	**36**
Max Cong	1.807	2.129	2.274	2.494
Time (sec)	88	200	195	207
Iterations	Limit	578	Limit	Limit
Exit Status	PDfeas. Iter: 302	ϵ-L-opt.	PDfeas. Iter: 700	PDfeas. Iter: 700

Sample computational experience

Table: *600 nodes, 990 arcs*

Iteration Limit: **300**, $\epsilon = 0.01$

	ΔB				
	10	**20**	**27**	**36**	**40**
obj	0.571562	1.076251	1.156187	1.088491	1.161887
sec	11848	7500	4502	11251	7800
Its	Limit	210	114	Limit	208
stat	PDfeas Iter: 300	ϵ-L-opt.	ϵ-L-opt.	PDfeas Iter: 300	ϵ-L-opt.

Sample computational experience

Table: *649 nodes, 1368 arcs,* $\Gamma(2)$

Iteration Limit: 500, $\epsilon = 0.01$

	ΔB		
	20	**30**	**40**
Max Cong	(0.06732) 1.294629	1.942652	(0.049348) 1.395284
Time (sec)	66420	36274	54070
Iterations	Limit	374	Limit
Exit Status	DF	ϵ-L-opt.	DF

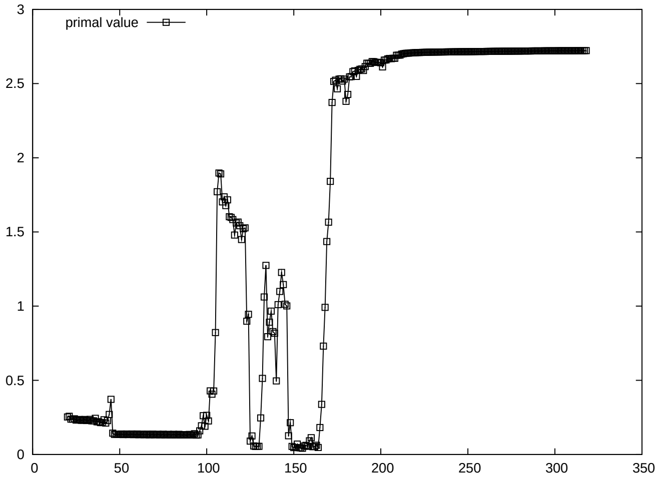
primal value
3
2.5
2
1.5
1
0.5
0
0
50
100
150
200
250
300
350

Table: Attack pattern

$x^u = 20$	$\Delta B = 57$	$x^u = 10$	$\Delta B = 27$	$x^u = 10$	$\Delta B = 36$
Range	**Count**	**Range**	**Count**	**Range**	**Count**
$[1, 1]$	8	$[1, 1]$	1	$[1, 1]$	14
$(1, 2]$	72	$(1, 2]$	405	$(1, 2]$	970
$(2, 3]$	4	$(2, 9]$	0	$(2, 5]$	3
$(5, 6]$	1	$(9, 10]$	3	$(5, 6]$	0
$(6, 7]$	1			$(6, 7]$	1
$(7, 8]$	4			$(7, 9]$	0
$(8, 20]$	0			$(9, 10]$	2

Impact

Ovl	Top 6 Arcs	R-3	R-3- 10%	C-all- 10%
2.15	29(7.79), 27(7.20), 41(7.03), 67(7.02), 54(6.72), 79(5.71)	1.718	1.335	1.671
1.79	29(8.28), 27(7.72), 41(7.32), 67(7.19), 54(6.92), 79(5.78)	1.431	1.112	1.386
1.56	29(8.31), 27(7.74), 41(7.53), 67(7.48), 54(7.18), 79(6.15)	1.227	0.953	1.213
1.36	29(8.18), 27(7.58), 41(7.53), 67(7.58), 54(7.22), 79(6.25)	1.073	0.834	1.055
1.20	29(8.43), 27(7.90), 41(7.53), 67(7.48), 54(7.18), 79(6.12)	0.954	0.741	0.936
1.08	29(7.87), 27(7.29), 41(7.04), 67(7.01), 54(6.70), 79(5.63)	0.859	0.668	0.839