Synchronization in Power Networks and in Non-uniform Kuramoto Oscillators

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Quick facts about the power grid:
- large-scale, complex, and nonlinear
- various dynamic phenomena and instabilities
- 100 years old and operating at its capacity limits
- increasing number of blackouts: New England '03, Italy '03, Brazil '09

Mathematical model of a power network:

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + P_{mi} - P_{ei}
\]

\[\theta(t)\] is measured w.r.t. a 60Hz rotating frame

Active node
Passive node
Reduce network to its active nodes
All-to-all admittance

Expected additional synergetic effects in future “smart grid”:
- increasing complexity and renewable stochastic power sources
- increasingly many transient disturbances to be detected and rejected

**Transient Stability:** Generators have to maintain synchronism in presence of large transient disturbances such as faults or loss of power lines and components, generation or load.

**The New York Times**


Energy is one of the top three national priorities [B. Obama, '09]

Classic model considered in transient stability analysis:

\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ji} \sin(\theta_i - \theta_j + \varphi_{ij})
\]
Mathematical model of a power network:
- swing equation for generator $i$:
  \[
  \frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + P_{mi} - P_{ei}
  \]
- $\theta(t)$ is measured w.r.t. a 60Hz rotating frame
- network-preserving model leads to DAEs
- network-reduction model leads to ODEs with reduced admittance matrix $Y_{ij} = |Y_{ij}| e^{i(\frac{\pi}{2} - \varphi_{ij})}$

$P_{ei} = E_i^2 G_{ii} + \sum_{j \neq i} E_i E_j |Y_{ij}| \sin(\theta_i - \theta_j + \varphi_{ij})$

**Classic model** considered in transient stability analysis:
\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})
\]

**Transient stability and synchronization:**
- frequency equilibrium: $(\dot{\theta_i}, \ddot{\theta_i}) = (0, 0)$ for all $i$
- synchronous equilibrium: $\theta_i - \theta_j$ bounded & $\dot{\theta}_i - \dot{\theta}_j = 0$ for all $\{i, j\}$

**Classical problem setup in transient stability analysis:**
- power network in stable frequency equilibrium
- $\rightarrow$ transient network disturbance and fault clearance
- stability analysis of a new frequency equilibrium in post-fault network

More general synchronization problem:
- synchronization in presence of transient network disturbances

**Classical analysis methods:** Hamiltonian arguments
\[
\frac{M_i}{\pi f_0} \ddot{\theta}_i = -D_i \dot{\theta}_i - \nabla_i U(\theta)^T
\]

Energy function analysis, (extended) invariance principle, analysis of reduced gradient flow [N. Kakimoto et al. '78, H.-D. Chiang et al. '94]
\[
\dot{\theta}_i = -\nabla_i U(\theta)^T
\]

Key objective: compute domain of attraction via numerical methods

**Open problem** [D. Hill and G. Chen '06]: power sys $\xrightarrow{?}$ network:
- transient stability, performance, and robustness of a power network $\xrightarrow{?}$ underlying network topology, parameters, and state
### Consensus Protocol in $\mathbb{R}^n$

\[ \dot{x}_i = - \sum_{j \neq i} a_{ij} (x_i - x_j) \]

- **Number of Identical Agents**: $n$ with state variable $x_i \in \mathbb{R}$
- **Graph**: with globally reachable node and weights $a_{ij} > 0$
- **Objective**: is state agreement: $x_i(t) - x_j(t) \to 0$
- **Application**: social networks, computer science, systems theory, robotic rendezvous, distributed computing, filtering and control . . .
- **Some References**: [M. DeGroot '74, J. Tsitsiklis '84, . . . ]

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### Kuramoto Model in $\mathbb{T}^n$

\[ \dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j) \]

- **Oscillators**: with phase $\theta_i \in \mathbb{T}$, frequency $\omega_i \in \mathbb{R}$, complete coupling
- **Objective**: is synchronization: $\theta_i(t) - \theta_j(t)$ bounded, $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
- **Application**: physics, biology, engineering, coupled neurons, Josephson junctions, motion coordination . . .
- **Some References**: [Y. Kuramoto '75, A. Winfree '80, . . . ]

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### Intro: The Big Picture

Open problem in synchronization and transient stability in power networks: relation to underlying network state, parameters, and topology

\[ \frac{M_i}{\pi f_0} \ddot{\theta}_i = - D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij}) \]

**Consensus Protocols**:

\[ \dot{x}_i = - \sum_{j \neq i} a_{ij} (x_i - x_j) \]

**Kuramoto Oscillators**:

\[ \dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j) \]
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From the swing equations to the Kuramoto model

\[ M_i \frac{\ddot{\theta}_i}{\pi f_0} = -D_i \dot{\theta}_i + \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \phi_{ij}) \]

\[ \dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j) \]

Possible connection has often been hinted at in the literature!

Power systems: [D. Subbarao et al., '01, G. Filatrella et al., '08, V. Fioriti et al., '09]
Networked control: [D. Hill et al., '06, M. Arcak, '07]
Dynamical systems: [H. Tanaka et al., '97]

Intro: The Big Picture

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Singular Perturbation Analysis

Time-scale separation in power network model:

- **Motivation:** harmonic oscillator

\[ \ddot{x} = -\frac{2}{\epsilon} \dot{x} - x \]

for \( \epsilon \ll 1 \Rightarrow \) two time-scales

- **Singular perturbation analysis:**

Outline

1. **Introduction**
   - synchronization and transient stability
   - power network model
   - consensus and Kuramoto oscillators

2. **Singular perturbation analysis**
   (to relate power network and Kuramoto model)

3. Synchronization analysis (of non-uniform Kuramoto model)
   - Main synchronization result
   - Sufficient condition (based on weakest lossless coupling)
   - Sufficient condition (based on lossless algebraic connectivity)
   - Further results

4. **Conclusions**
Time-scale separation in power network model:

- **Motivation:** harmonic oscillator
  \[ \ddot{x} = -\frac{2}{\epsilon} \dot{x} - x \]

- **Singular perturbation analysis:**
  \[ \epsilon \ll 1 \quad \epsilon < 1 \quad \epsilon = 1 \]
  \[ \epsilon \gg 1 \]
  for \( \epsilon \ll 1 \rightarrow \) two time-scales

\[ z(0) \neq h(x(0)) \]

- **Discussion** of the assumption \( \epsilon = \frac{M_{\max}}{\pi f_0 D_{\min}} \) sufficiently small:
  - physical interpretation: damping and sync on separate time-scales
  - classic assumption in literature on coupled oscillators: over-damped mechanical pendula and Josephson junctions
  - physical reality: with generator internal control effects \( \epsilon \in O(0.1) \)
  - simulation studies show accurate approximation even for large \( \epsilon \)
  - non-uniform Kuramoto model corresponds to reduced gradient system \( \dot{\theta}_i = -\nabla_i U(\theta)^T \) used successfully in academia and industry since 1978

Tikhonov’s Theorem:
Assume the non-uniform Kuramoto model synchronizes exponentially. Then \( \forall (\theta(0), \dot{\theta}(0)) \) there exists \( \epsilon^* > 0 \) such that \( \forall \epsilon < \epsilon^* \) and \( \forall t \geq 0 \)
\[ \theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = O(\epsilon). \]
Main Synchronization Result

Conditions on network parameters:

network connectivity > network's non-uniformity + network's losses, and gap determines domain of attraction

1. Non-Uniform Kuramoto Model:
   ⇒ exponential synchronization: phase locking & frequency entrainment
   ⇒ for $\varphi_{ij} = 0$: explicit synchronization frequency & synchronization rates
   ⇒ for $\varphi_{ij} = 0$ & $\omega_i = \omega_j$: exponential phase synchronization

2. Power Network Model:
   ⇒ there exists $\epsilon$ sufficiently small such that for all $t \geq 0$
   \[ \theta_i(t)_{\text{power network}} - \theta_i(t)_{\text{non-uniform Kuramoto model}} = O(\epsilon). \]
   ⇒ for $\epsilon$ and network losses $\varphi_{ij}$ sufficiently small, $O(\epsilon)$ error converges

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Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$:

$$D_i \dot{\theta}_i = \omega_i - \sum_{j \neq i} P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij})$$

- **Non-uniformity** in network: $D_i, \omega_i, P_{ij}, \varphi_{ij}$
- **Directed coupling** between oscillator $i$ and $j$:
  - coupling weights: $\frac{P_{ij}}{D_i} \neq \frac{P_{ji}}{D_j}$
  - coupling functions: $\sin(\theta_i - \theta_j + \varphi_{ij}) + \sin(\theta_j - \theta_i + \varphi_{ij}) \neq 0$
- **Phase shift** $\varphi_{ij}$ induces lossless and lossy coupling:
  $$P_{ij} \sin(\theta_i - \theta_j + \varphi_{ij}) = P_{ij} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + P_{ij} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

**Synchronization analysis** in multiple steps:

1. phase locking: $\theta_i(t) - \theta_j(t)$ becomes bounded
2. frequency entrainment: $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$
3. phase synchronization: $\dot{\theta}_i(t) - \dot{\theta}_j(t) \to 0$

Synchronization of Non-Uniform Kuramoto Oscillators

Classic (uniform) Kuramoto Model in $\mathbb{T}^n$:

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \sum_{j \neq i} \sin(\theta_i - \theta_j)$$

**Condition (1) for synchronization:**

$$K > \omega_{\text{max}} - \omega_{\text{min}}$$

Gap determines the admissible initial lack of phase locking in a $\frac{\pi}{2}$ interval.

Condition (1) strictly improves existing bounds on Kuramoto model: [F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, A. Jadbabaie et al. '04, J.L. van Hemmen et al. '93].

Necessary condition for sync of $n$ oscillators: $K > \frac{n}{2(n-1)} (\omega_{\text{max}} - \omega_{\text{min}})$ [J.L. van Hemmen et al. '93, A. Jadbabaie et al. '04, N. Chopra et al. '09]

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Synchronization of Non-Uniform Kuramoto Oscillators

Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

**Condition (1) for synchronization:**

Assume the graph induced by $P = P^T$ is complete and

$$\frac{n P_{\text{min}}}{D_{\text{max}} \cos(\varphi_{\text{max}})} > \frac{\max_{i,j} (\omega_i - \omega_j)}{\max_{i,j} (\frac{\omega_i}{D_i} - \frac{\omega_j}{D_j})} + \max_i \sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij})$$

worst lossless coupling  worst non-uniformity  worst lossy coupling

Gap determines the admissible initial lack of phase locking in a $\frac{\pi}{2}$ interval.

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Theorem: Phase locking and frequency entrainment (1)

Non-uniform Kuramoto with complete $P = P^T$

Assume minimal coupling larger than a critical value, i.e.,

$$P_{\text{min}} > P_{\text{critical}} := \frac{D_{\text{max}}}{n \cos(\varphi_{\text{max}})} \left( \max_{i,j} \left( \frac{\omega_i}{D_i} - \frac{\omega_j}{D_j} \right) + \max_i \sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \right)$$

Define $\delta = \frac{\pi}{2} - \arccos(\cos(\varphi_{\text{max}}) \frac{P_{\text{critical}}}{P_{\text{min}}})$ and set of locked phases

$$\Delta(\delta) := \{ \theta \in \mathbb{T}^n | \max_{i,j} |\theta_i - \theta_j| \leq \delta \}$$

Then

1. **phase locking**: the set $\Delta(\delta)$ is positively invariant
2. **frequency entrainment**: $\forall \theta(0) \in \Delta(\delta)$ the frequencies $\dot{\theta}_i(t)$ synchronize exponentially to some frequency $\dot{\theta}_\infty \in [\dot{\theta}_{\text{min}}(0), \dot{\theta}_{\text{max}}(0)]$
Non-uniform Kuramoto Model in $\mathbb{T}^n$ - rewritten:

$$\dot{\theta}_i = \frac{\omega_i}{D_i} - \sum_{j \neq i} \frac{P_{ij}}{D_i} \cos(\varphi_{ij}) \sin(\theta_i - \theta_j) + \frac{P_{ij}}{D_i} \sin(\varphi_{ij}) \cos(\theta_i - \theta_j)$$

**Condition (2) for synchronization:**

Assume the graph induced by $P = P^T$ is connected with unweighted Laplacian $L$ and weighted Laplacian $L(P_{ij} \cos(\varphi_{ij}))$ and

$$\lambda_2(L(P_{ij} \cos(\varphi_{ij}))) > f(D_i) \cdot \left(1/\cos(\varphi_{\max})\right) \times$$

lossless connectivity non-uniform $D_i$s necessary phase locking

$$\left(\left\|\left[\ldots, \frac{\omega_i}{D_i}, \ldots\right]\right\|_2 + \sqrt{\lambda_{\text{max}}(L)} \left\|\left[\ldots, \sum_j \frac{P_{ij}}{D_i} \sin(\varphi_{ij}), \ldots\right]\right\|_2\right)$$

non-uniformity lossy coupling

Gap determines the admissible initial lack of phase locking in a $\pi$ interval.

**Condition (2) for synchronization:**

$$K > \left\|\left[\ldots, \omega_i - \omega_j, \ldots\right]\right\|_2$$

Gap determines the admissible initial lack of phase locking in a $\pi$ interval.

Condition (2) corresponds to the bound in [N. Chopra et al. ’09].
**Synchronization of Non-Uniform Kuramoto Oscillators**

**Theorem: Phase locking and frequency entrainment (2)**

Assume graph induced by \( P = P^T \) is connected with unweighted Laplacian \( L \), incidence matrix \( H \), and weighted Laplacian \( L(P_{ij} \cos(\varphi_{ij})) \).

Assume algebraic connectivity is larger than a critical value, i.e.,

\[
\lambda_2(L(P_{ij} \cos(\varphi_{ij}))) > \lambda_{\text{critical}} := \left\| HD^{-1}\omega \right\|_2 + \sqrt{\lambda_{\text{max}}(L) \left[ \ldots \sum_{ij} P_{ij} \sin(\varphi_{ij}) \ldots \right]} \right\|_2 \cos(\varphi_{\text{max}})/(\kappa/n) \min_{i,j} \{ D_{\neq(i,j)} \},
\]

where \( \kappa := \sum_{k=1}^n \frac{1}{D_{\neq k}} \), \( \mu := \sqrt{\min_{i,j} \{ D(D_i) \} / \max_{i,j} \{ D(D_j) \} } \).

Define \( \phi_{\text{min}} \in (0, \frac{\pi}{2}) \) by \( \text{sinc}(\pi - \phi_{\text{min}}) = (2/\pi)\lambda_{\text{critical}}/\lambda_2(L(P_{ij} \cos(\varphi_{ij}))) \).

1) **Phase locking:** \( \forall \left\| H\theta(0) \right\|_2 \leq \mu(\pi - \phi_{\text{min}}) \), there is \( T \geq 0 \) such that \( \left\| H\theta(t) \right\|_2 < \frac{\pi}{2} - \varphi_{\text{max}} \) for all \( t > T \)

2) **Frequency entrainment:** \( \forall \left\| H\theta(0) \right\|_2 \leq \mu(\pi - \phi_{\text{min}}) \) the frequencies \( \dot{\theta}_i(t) \)

synchronize exponentially to some frequency \( \dot{\theta}_\infty \in [\dot{\theta}_{\text{min}}(0), \dot{\theta}_{\text{max}}(0)] \)

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**Theorem: A refined result on frequency entrainment**

Assume graph induced by \( P \) has globally reachable node and there exists \( \delta \in (0, \frac{\pi}{2}) \) such that the phases are locked in the set \( \Delta(\delta) \)

If \( P = P^T \) & \( \varphi_{ij} = 0 \) for all \( i, j \in \{1, \ldots, n\} \), then \( \forall \theta(0) \in \Delta(\delta) \) the frequencies \( \dot{\theta}_i(t) \) synchronize exp. to the weighted mean frequency

\[
\Omega_c = \frac{1}{\sum_i D_i} \sum_i D_i \omega_i
\]

and the exponential synchronization rate is no worse than

\[
\lambda_{\text{fe}} = -\frac{\lambda_2(L(P_{ij})) \sin(\delta) \cos(\angle(D(1), \mathbf{1}))^2}{D_{\text{max}}} \frac{1}{\angle(D(1), \mathbf{1})}
\]

where

- \( \lambda_{\text{fe}} \) is the fastest exponent for frequency entrainment
- \( \lambda_2 \) is the second smallest eigenvalue
- \( D_{\text{max}} \) is the maximum degree of the graph
- \( \angle(D(1), \mathbf{1}) \) is the angle between the vector \( D(1) \) and the vector \( \mathbf{1} \)
Synchronization of Non-Uniform Kuramoto Oscillators

Theorem: A result on phase synchronization

Assume the graph induced by $P$ has a globally reachable node, and $\varphi_{ij} = 0$ and $\frac{\omega_i}{D_i} = \frac{\omega_j}{D_j}$ for all $i, j \in \{1, \ldots, n\}$. Let $\phi \in (0, \pi]$.

For the non-uniform Kuramoto model,

1) For all $\theta(0) \in \{\theta \in \mathbb{T}^n : \max_{i,j} |\theta_i - \theta_j| < \pi - \phi\}$ the phases $\theta_i(t)$ synchronize exponentially; and

2) if $P = P^T$, $\forall \|H\theta(0)\|_2 \leq \mu(\pi - \phi)$ the phases $\theta_i(t)$ synchronize exponentially at a rate no worse than

$$\lambda_{ps} = -\left(\frac{\kappa}{n}\min_{i,j \neq (i,j)}\left\{D_i\right\}\cdot \frac{\sin(\pi - \phi)}{\theta(0)}\cdot \lambda_2(L(P_{ij}))\right)$$

weight of $D_i$ connectivity

Result can be reduced to [A. Jadbabaie et al. '04].

Simulation Studies

Simulation data:
- Initial phases mostly clustered besides red phasor
- $\epsilon = 0.6s$ is large
- Non-uniform network

Result: singular perturbation analysis is accurate ✓ both models synchronize ✓

Summary:
Open problem in synchronization and transient stability in power networks:
- Relation to underlying network state, parameters, and topology (not today)
- Time-varying Consensus Protocols:
- Singular Perturbation Approximation
- Non-uniform Kuramoto Oscillators

Future Work:
- Relation to network topology, clustering and scalability
- Synchronization in optimal power flow problems